momentum, and turbulence energy. The set of equations includes three constants whose values were taken as: $h_s = 0.932$ (which gives $h_s^{3/2} = 0.9$), $\sigma_k = 2.0$, and $C_D = 0.3$ for flows where $\epsilon \ll U$, and 0.08 for flows with ϵ of the same order as U. Different values reported in the literature for the last two constants are quoted by Launder and Spalding,[1] ranging from 0.06 to 0.1 for C_D and from 1.0 to 2.0 for σ_k.

Predictions were obtained initially with the value of λ selected for each flow to offer the best possible agreement with experiment. The selected values are 0.0464 for flow 1, 0.105 for flow 2, 0.095 for flow 3, and 0.08 for flow 4. These values are labeled λ_s and are in fair agreement with the literature, where a value of 0.04 was used by Emmons[2] for the plane jet, and of 0.06 was used by Sechagiri[3] for the plane interacting wakes. Figures 2 and 3 present prediction results obtained using the λ equation. Figure 2 shows the streamwise variations of the calculated λ for each flow in relation to λ_s. It can be noticed that the value of (λ/λ_s) is close to unity for all flows if we discard values obtained for the first few steps of the marching integration calculation, where results are much influenced by the accuracy of initial conditions. Although values of λ_s vary from 0.0464 to 0.105 for flows considered, the maximum deviation of (λ/λ_s) from unity is about 20%.

Figure 3 shows predictions of the streamwise variations of the following characteristics for each flow: maximum velocity difference across the flow ϵ, maximum value of turbulent kinetic energy k_m, and boundary-layer thickness $\delta_{1/2}$. In general, agreement between predictions and experiment is satisfactory for all flows considered. This agreement is generally better for mean velocity values as compared to values of turbulent kinetic energy, which is well expected regarding the larger inaccuracies associated with the experimental measurements of the later quantity and the amount of empirical input in its conservation equation.

Conclusions

The semiempirical relation obtained for the length-scale of turbulence has potentially improved the universality of the one-equation model. With the same values of empirical constants, the model provided acceptable predictions of both mean and turbulent quantities for four flows with different geometries. The model might therefore be recommended for the prediction of two-dimensional free shear flows. The advantage of the two-equation model over the present model is not expected to be significant enough to justify solving one extra partial differential equation and dealing with five or six empirical constants instead of three.

Acknowledgment

The author would like to acknowledge many helpful discussions with W.B. Nicoll and E. Brundrett during the course of this investigation, which was financially supported by the National Research Council of Canada.

References

[1] Launder, B.E. and Spalding, D.B., "Mathematical Models of Turbulence," Academic Press, New York, 1972.

[2] Emmons, H.W., "Shear Flow Turbulence," *Proceedings of the 2nd U.S. Congress of Applied Mechanics*, ASME, 1954.

[3] Sechagiri, B.V., "Properties of Turbulent Interacting Wakes," Ph.D. Thesis, University of Waterloo, 1971.

[4] Patankar, S.V. and Spalding, D.B., *Heat and Mass Transfer in Boundary Layers*, Morgan-Grampian Press, London, 1967.

[5] Spalding, D.B., "The Calculation of the Length Scale of Turbulence in Some Shear Flows Remote from Walls," *Progress in Heat and Mass Transfer*, Vol. 2, 1969, p. 255.

[6] Heskestad, G., "Hot Wire Measurements in a Plane Turbulent Jet," *Journal of Applied Mechanics*, Vol. 4, 1965, p. 721.

[7] Townsend, A.A., "The Structure of Turbulent Shear Flow," Cambridge University Press, New York, 1956.

[8] Naudascher, E., "Flow in the Wake of Self-Propelled Bodies and Related Sources of Turbulence," *Journal of Fluid Mechanics*, Vol. 22, 1965, p. 625.

[9] Zawacki, T.S. and Weinstein, H., "Experimental Investigation of Turbulence in the Mixing Region Between Coaxial Streams," NASA CR-959, 1968.

Derivation of an Integral Equation for Three-Dimensional Transonic Flows

Wandera Ogana*
University of Nairobi, Nairobi, Kenya

Introduction

HEASLET and Spreiter[1] derived the integral equation for steady, inviscid, three-dimensional transonic flows. They stated that the field point was surrounded by a sphere of infinitesimal radius, but in the subsequent analysis they actually surrounded the field point by an infinite strip of infinitestimal thickness in the streamwise direction. In this Note, we surround the field point by a hexahedron and derive an integral equation from which the equations by Heaslet and Spreiter[1] can be deduced as a special case. The method is similar to that used by Ogana and Spreiter[2] to derive the integral equation for two-dimensional transonic flows. Although numerical methods are not analyzed here, the approach used by Ogana[3] can be applied to obtain numerical solutions to the integral equation derived by the present method.

Mathematical Preliminary

The mathematical concepts essential to the analysis are briefly presented in this section.

1) Consider the question of convergence of a volume integral. Let f be a function which becomes infinite at a point P within the volume V, then f is singular at P. To define the integral of f throughout the volume V, we surround the singular point P by a small enclosed cavity σ, then let σ vanish while always surrounding P, that is[4]:

$$\int_V f \mathrm{d}V \equiv \lim_{\sigma \to 0} \int_{V-\sigma} f \mathrm{d}V \tag{1}$$

The volume integral is said to be convergent if the limit on the right-hand side of Eq. (1) is finite; otherwise, it is divergent. If the integral is convergent, but the value of the limit depends on the shape of σ, it is said to be semiconvergent.[4]

Convergence of the volume integral is guaranteed if within a sphere of finite radius, whose center is P, the function f satisfies the inequality:

$$|f| < Mr^{-\mu} \tag{2}$$

where $\mu < 3$, M is a definite constant, and r is the distance between P and the point at which f is estimated. For divergence, $\mu \geq 3$, but semiconvergence may occur if $\mu = 3$.

2) Now, consider Green's theorem when some of the integrals may be semiconvergent and also consider the differentiation of a volume integral with respect to a parameter. Let the function Ω, together with its first- and second-order derivatives, be finite throughout the volume V. In addition, let ψ be a function which becomes infinite at a point P in the volume V. Green's theorem,[4] applied to the region bounded internally by σ and externally by a surface S, is:

$$\int_V (\psi \nabla^2 \Omega - \Omega \nabla^2 \psi) \mathrm{d}V = -\int_S \left(\psi \frac{\partial \Omega}{\partial n} - \Omega \frac{\partial \psi}{\partial n} \right) \mathrm{d}S - \lim_{\sigma \to 0} \int_\sigma \left(\psi \frac{\partial \Omega}{\partial n} - \Omega \frac{\partial \psi}{\partial n} \right) \mathrm{d}\sigma \tag{3}$$

Received July 10, 1978.
Index categories: Subsonic Flow; Transonic Flow.
*Lecturer, Department of Mathematics. Member AIAA.

where ∇^2 is the Laplacian operator and n is the inward normal at the boundary. The surface S includes the far-field boundary, in infinite domains, and surfaces on which either Ω or ψ is discontinuous.

Let $\tau=\tau(\xi,\eta,\zeta)$ be finite and have uniformly continuous derivatives in the region V and let $g=g(\xi-x,\ \eta-y,\ \zeta-z)$ be infinite at the point P whose coordinates are (x,y,z). The following result can be derived[4]:

$$\frac{\partial}{\partial x}\left[\int_V \tau(\xi,\eta,\zeta)g(\xi-x,\ \eta-y,\ \zeta-z)\mathrm{d}V\right]$$

$$=\int_V \tau(\xi,\eta,\zeta)\frac{\partial}{\partial x}\left[g(\xi-x,\ \eta-y,\ \zeta-z)\right]\mathrm{d}V$$

$$-\tau(x,y,z)\lim_{\sigma\to 0}\left[\int_\sigma g(\xi-x,\ \eta-y,\ \zeta-z)n_x\mathrm{d}\sigma\right] \tag{4}$$

where $\mathrm{d}V=\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$ and $n_x=-\cos(n,x)$.

Integral Equation

For subsequent convenience of notation, we let $\bar{x}$, $\bar{y}$, and $\bar{z}$ be the Cartesian coordinates in the physical space. The freestream is taken in the $\bar{x}$ direction and the wing in the $\bar{x}\bar{y}$ plane. The transonic small-disturbance equation is:

$$\beta^2\bar{\phi}_{\bar{x}\bar{x}}+\bar{\phi}_{\bar{y}\bar{y}}+\bar{\phi}_{\bar{z}\bar{z}}=M_\infty^2(\gamma+1)\bar{\phi}_{\bar{x}}\bar{\phi}_{\bar{x}\bar{x}} \tag{5}$$

where $\beta^2=1-M_\infty^2$, $\bar{\phi}$ is the perturbation velocity potential, M_∞ is the freestream Mach number, and γ is the ratio of specific heats.

The perturbation velocity components are given by:

$$\bar{u}=\bar{\phi}_{\bar{x}},\quad \bar{v}=\bar{\phi}_{\bar{y}},\quad \bar{w}=\bar{\phi}_{\bar{z}} \tag{6}$$

The transformations:

$$x=\bar{x},\ y=\beta\bar{y},\ z=\beta\bar{z},\ \phi(x,y,z)=\frac{(\gamma+1)M_\infty^2}{\beta^2}\bar{\phi}(\bar{x},\bar{y},\bar{z}) \tag{7}$$

convert Eq. (5) to:

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=\phi_x\phi_{xx} \tag{8}$$

The transformed velocity components are

$$u=\phi_x,\quad v=\phi_y,\quad w=\phi_z \tag{9}$$

We consider transonic flows for which $M_\infty<1$ and apply Green's theorem, Eq. (3), by identifying Ω with ϕ and ψ with the elementary solution of $\nabla^2\psi=0$; that is,

$$\psi(\xi-x,\ \eta-y,\ \zeta-z)=\frac{1}{[(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2]^{1/2}} \tag{10}$$

Since ψ is singular at the point $P(x,y,z)$, we surround P by a vanishing cavity σ. For the present analysis, σ is chosen to be the hexahedron shown in Fig. 1. The quantity λ (Fig. 1) is a definite constant; hence, the vanishing of the cavity σ corresponds to $\epsilon\to 0$. Green's theorem yields

$$-\lim_{\epsilon\to 0}\int_\sigma\left(\psi\frac{\partial\phi}{\partial n}-\phi\frac{\partial\psi}{\partial n}\right)\mathrm{d}\sigma=\int_{S_w+S_\Sigma+S_V}\left(\psi\frac{\partial\phi}{\partial n}-\phi\frac{\partial\psi}{\partial n}\right)\mathrm{d}S$$

$$+\int_V\psi\frac{\partial}{\partial\xi}\left[\frac{u^2(\xi,\eta,\zeta)}{2}\right]\mathrm{d}V \tag{11}$$

where we have used Eq. (8) and $\nabla^2\psi=0$. S_w denotes the wing surface, S_Σ the shock, and S_V the trailing vortex sheet.

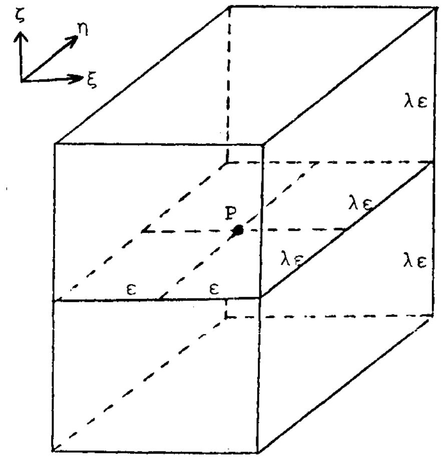

Fig. 1 Location of the field point P in a hexahedron of infinitesimal dimensions ($\epsilon\to 0$).

As $\epsilon\to 0$, ϕ and its derivatives assume values at P; hence, the integral around the cavity becomes

$$\phi(x,y,z)\lim_{\epsilon\to 0}\int_\sigma\frac{\partial\psi}{\partial n}\mathrm{d}\sigma \tag{12}$$

Let I_1 be the contribution to the integral from the faces of the hexahedron which lie in the yz plane and I_2 be the contribution from the faces which lie in the other planes. It can be shown that:

$$I_1=-8\arctan[\lambda^2(1+2\lambda^2)^{-1/2}] \tag{13a}$$

$$I_2=-16\arctan(1+2\lambda^2)^{-1/2} \tag{13b}$$

After using some trigonometric identities, we obtain

$$\phi(x,y,z)\lim_{\epsilon\to 0}\int_\sigma\frac{\partial\psi}{\partial n}\mathrm{d}\sigma=(I_1+I_2)\phi(x,y,z)=-4\pi\phi(x,y,z) \tag{14}$$

Equation (11) becomes

$$\phi(x,y,z)=-\frac{1}{4\pi}\int_{S_w+S_\Sigma+S_V}\left(\psi\frac{\partial\phi}{\partial n}-\phi\frac{\partial\psi}{\partial n}\right)\mathrm{d}S$$

$$-\frac{1}{8\pi}\int_V\psi\frac{\partial}{\partial\xi}[u^2(\xi,\eta,\zeta)]\mathrm{d}V \tag{15}$$

If the volume integral is integrated by parts, this equation reduces to[1,5]

$$\phi(x,y,z)=\phi_B(x,y,z)+\frac{1}{8\pi}\int_V u^2(\xi,\eta,\zeta)\psi_\xi\mathrm{d}V \tag{16}$$

where

$$\phi_B(x,y,z)=\int_{S_w}\left(\frac{\partial\psi}{\partial\zeta}\Delta\phi-\psi\Delta w\right)\mathrm{d}S+\int_{S_V}\frac{\partial\psi}{\partial\zeta}\Delta\phi\mathrm{d}S$$

and where $\Delta\phi$ is the jump in potential across the wing surface and Δw is related to the wing thickness distribution. The term $\phi_B(x,y,z)$ has been considered in greater detail by Klunker.[5]

The integral equation in u is obtained by differentiating a singular volume integral with respect to a parameter. In Eq. (4), we identify τ with u^2 and g with ψ_ξ, hence,

$$\frac{\partial}{\partial x}\left[\int_V u^2\psi_\xi\mathrm{d}V\right]=\int_V u^2\psi_{\xi x}\mathrm{d}V-u^2(x,y,z)$$

$$\times\lim_{\epsilon\to 0}\left[\int_\sigma\psi_\xi(\xi-x,\ \eta-y,\ \zeta-z)n_x\mathrm{d}\sigma\right] \tag{17}$$

The surface integral around σ is evaluated only on the two surfaces of the hexahedron which lie in the yz plane because $n_x=0$ on all the other surfaces. By using the change of variables $\eta-y=s\epsilon$ and $\zeta-z=t\epsilon$, it can be shown that:

$$\lim_{\epsilon\to 0}\int_\sigma \psi_\xi(\xi-x,\ \eta-y,\ \zeta-z)n_x\mathrm{d}\sigma$$

$$=-2\int_{-\lambda}^{\lambda}\int_{-\lambda}^{\lambda}\frac{\mathrm{d}s\mathrm{d}t}{(1+s^2+t^2)^{3/2}}$$

$$=-8\arctan[\lambda^2(1+2\lambda^2)^{-1/2}] \tag{18}$$

If we differentiate Eq. (16) with respect to x and use Eqs. (17 and 18), we obtain the integral equation:

$$u(x,y,z)=u_B(x,y,z)+\nu u^2(x,y,z)$$

$$+\int_V K(\xi-x,\ \eta-y,\ \zeta-z)u^2(\xi,\eta,\zeta)\mathrm{d}V \tag{19}$$

where

$$u_B=\partial\phi_B/\partial x \tag{20a}$$

$$\nu=(1/\pi)\arctan[\lambda^2(1+2\lambda^2)^{-1/2}] \tag{20b}$$

$$K(\xi-x,\ \eta-y,\ \zeta-z)=\frac{(\eta-y)^2+(\zeta-z)^2-2(\xi-x)^2}{8\pi[(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2]^{5/2}} \tag{20c}$$

The volume integral in Eq. (19) is defined according to Eq. (1) because the integrand is infinite at (x,y,z). If u is finite everywhere in the region V, then the use of spherical coordinates $\xi-x=r\sin\theta\cos\phi$, $\eta-y=r\sin\theta\ \sin\phi$, and $\zeta-z=r\ \cos\theta$ shows that $|Ku^2|<Mr^{-3}$ where $|u^2/4\pi|\leq M$. Hence, Eq. (2) is satisfied with $\mu=3$. This indicates that if the volume integral in Eq. (19) converges, then it must be semiconvergent. The value of each of the last two terms in Eq. (19) depends on the shape of the cavity σ, hence on λ. But their sum is independent of λ, because it represents the derivative of a convergent integral.

The integral equation derived by Heaslet and Spreiter[1] can be obtained by taking the limit as $\lambda\to\infty$ to get $\nu=1/2$. Surrounding the singularity by a vanishing sphere is equivalent to surrounding it by a vanishing cube.[6] The resulting integral equation is deduced by choosing $\lambda=1$ so that $\nu=1/6$.

There arises a situation similar to what occurs for the two-dimensional flow[2]; namely, by varying the value of λ, it appears that we are getting different integral equations. In reality, the integral equations are the same provided we define the volume integral in Eq. (19) according to Eq. (1) and use the same cavity around which Eq. (18) is evaluated.

References

[1] Heaslet, M. A. and Spreiter, J. R., "Three-Dimensional Transonic Flow Theory Applied to Slender Wings and Bodies," NACA Rept. 1318, 1957.

[2] Ogana, W. and Spreiter, J. R., "Derivation of an Integral Equation for Transonic Flows," *AIAA Journal*, Vol. 15, Feb. 1977, pp. 281-283.

[3] Ogana, W., "Numerical Solution for Subcritical Flows by a Transonic Integral Equation Method," *AIAA Journal*, Vol. 15, March 1977, pp. 444-446.

[4] Leathem, J. G., *Volume and Surface Integrals Used in Physics*, Hafner Publishing Co., New York, 1960, pp. 1-29.

[5] Klunker, E. B., "Contribution to Methods for Calculating the Flow About Thin Lifting Wings at Transonic Speeds—Analytic Expressions for the Far Field," NASA TN D-6530, 1971.

[6] Ledermann, W., *Multiple Integrals*, Dover Publications, New York, 1966, pp. 56-58.

Overexpanded Two-Dimensional Transonic Free Jet

R. D. Small*

University of California, Los Angeles, Los Angeles, Calif.

Nomenclature

G	= Green's function
H	= nozzle exit half height
i	$=\sqrt{-1}$
K	= transonic similarity parameter
ℓ	= integration path
M	= Mach number
m	= index
n	= denotes normal to boundary
p	= pressure
t	= dummy variable
U	= velocity
X, Y, y	= Cartesian coordinates
x	= transonic coordinate $=X/\sqrt{\epsilon}H$
γ	= ratio of specific heats
ϵ	= perturbation parameter $=(U_e-U_\infty)/U_\infty$
ξ,η	= dummy variables
Φ	= exact potential
ϕ	= perturbation potential

Subscripts

e	= nozzle exit plane
s	= denotes shock wave surface
∞	= conditions on jet boundaries

Superscript

*	= locally sonic conditions

I. Introduction

COMPLICATED flow patterns develop when an overexpanded transonic nozzle jet flow adjusts to a high-back pressure. The shock wave patterns that form depend on the exit pressure, Mach number, and far-field pressure. In general, two basic regimes can be distinguished. The first involves supersonic flow on the boundaries and the flow adjustment occurs through either regular reflection of an oblique shock wave or through Mach reflection. In both cases, regions of embedded subsonic flow can occur.[1] For exit Mach numbers less than 1.486 ($\gamma=1.4$), it is not certain[2] that Mach reflection occurs for two-dimensional flows.

When the pressure ratio p_e/p_∞ is sufficiently high, subsonic flow occurs on the jet boundaries and a strong curved shock stands at the nozzle exit. The flowfield bounded by the curved shock waves, jet boundaries, and downstream infinity is entirely subsonic and, hence, elliptic. In contrast to jet flows with supersonic boundaries, calculations of the flowfield with subsonic boundaries must be carried out to downstream infinity, at least in the inviscid approximation.

In the transonic regime, shock waves can be considered isentropic as entropy changes are $0[(M_e^2-1)^3]$ and for exit Mach numbers sufficiently close to one, a small disturbance approximation to the full-potential equation developed. For subsonic boundaries, a far-field approximation for the flow at

Received July 13, 1978.

Index categories: Transonic Flow; Jets, Wakes, and Viscid-Inviscid Flow Interactions.

*Visiting Asst. Prof., Mechanics & Structures Department; currently on leave from the Dept. of Aeronautical Engineering., Technion, Israel Institute of Technology, Haifa, Israel. Member AIAA.